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## FALL 2014 NU PUTNAM SELECTION TEST

**Problem A1.** Show that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(1+n)} \leq \pi \, .$$

- *Answer:* The sum is bounded by

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(1+x)} = 2 \int_0^{\infty} \frac{dy}{1+y^2} = \pi \, .$$

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**Problem A2.** Find the following infinite product:

$$P = \prod_{n=1}^{\infty} \left(1 + (\tfrac{1}{7})^{2^n}\right)$$

Write the result as a fraction $P = \frac{a}{b}$ in least terms.

- *Answer:* Let $P_N = \prod_{n=1}^{N} \left(1 + (\frac{1}{7})^{2^n}\right)$. Then, we have

$$\begin{aligned}
\left(1 - (\tfrac{1}{7})^2\right) P_N &= \left(1 - (\tfrac{1}{7})^2\right) \left(1 + (\tfrac{1}{7})^2\right) \left(1 + (\tfrac{1}{7})^4\right) \left(1 + (\tfrac{1}{7})^8\right) \cdots \left(1 + (\tfrac{1}{7})^{2^N}\right) \\
&= \left(1 - (\tfrac{1}{7})^4\right) \left(1 + (\tfrac{1}{7})^4\right) \left(1 + (\tfrac{1}{7})^8\right) \cdots \left(1 + (\tfrac{1}{7})^{2^N}\right) \\
&= \left(1 - (\tfrac{1}{7})^8\right) \left(1 + (\tfrac{1}{7})^8\right) \cdots \left(1 + (\tfrac{1}{7})^{2^N}\right) \\
&\cdots \\
&= \left(1 - (\tfrac{1}{7})^{2^{(N+1)}}\right) .
\end{aligned}$$

Hence,

$$\left(1 - (\tfrac{1}{7})^2\right) P = \lim_{N\to\infty} \{\left(1 - \tfrac{1}{7}\right) P\} = \lim_{N\to\infty} \left(1 - (\tfrac{1}{7})^{2^{(N+1)}}\right) = 1\,,$$

and $P = \dfrac{1}{1 - (\frac{1}{7})^2} = \dfrac{49}{48}$.

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**Problem A3.** Let S be a set with even number of elements, and $f : S \to S$ a map of S into itself such that $f \circ f : S \to S$ is the identity map. Show that the set of the fixed points has even number of elements.

- *Answer:* Let $T$ be the set of points $x \in S$ such that $f(x) \neq x$ (non-fixed points). Consider the set $P$ of unordered pairs $\{z, f(z)\}$ with $z \in T$. The map $\theta : T \to P$ defined by $\theta(x) = \{x, f(x)\}$ is exactly two-to-one: the only preimage of $\{z, f(z)\}$ are the two distinct elements $z$ and $f(z)$. It follows that $|T| = 2|P|$, i.e., $T$ has even number of elements. Its complement $S \setminus T$ also has even number of elements.

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**Problem A4.** Let $f : \mathbb{R} \to \mathbb{R}$ a continuous function without fixed points, i.e., there is no $x \in \mathbb{R}$ such that $f(x) = x$. Let $n$ be a positive integer. Prove that $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n}$ has no fixed points either.

- *Answer:* If $f$ has not fixed points then $f(x) - x$ is never zero, so either $f(x) - x > 0$ for every $x \in \mathbb{R}$, or $f(x) - x < 0$ for every $x \in \mathbb{R}$. Then for every $x \in \mathbb{R}$ the sequence $x, f(x), f^2(x), \dots$ is strictly increasing or strictly decreasing, and consequently we cannot have $f^n(x) = x$.

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**Problem A5.** The Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, \dots$ are defined as $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). The *digital root* of a non-negative integer is the (single digit) value obtained by an iterative process of summing digits, on each iteration using the result from the previous iteration to compute a digit sum. The process continues until a single-digit number is reached. For example, the digital root of 65,536 is 7, because $6 + 5 + 5 + 3 + 6 = 25$ and $2 + 5 = 7$. Prove that there are integers $a, b$, with $a > 0$ and $b \geq 0$, such that all Fibonacci numbers of the form $F_{an+b}$, $n = 0, 1, 2, 3, \dots$, have the same digital root.

- *Answer:* The digital root of a positive number is just its residue modulo 9, except when the residue is zero, in which case the digital sum is 9. So all we need to prove is that for some $a > 0$, $b \geq 0$, $F_{an+b}$ is constant modulo 9. For each $k \geq 0$, let $f_k$ the integer in $[0, 1, 2, 3, 4, 5, 6, 7, 8]$ such that $F_k \equiv f_k \pmod 9$. Note that $f_k + f_{k+1} \equiv f_{k+2} \pmod 9$.

Since the set of pairs $(p, q)$, $0 \leq p, q \leq 8$, is finite, the sequence of pairs $(f_k, f_{k+1})$ will end up being the same for two different values of $k < k'$: $(f_k, f_{k+1}) = (f_{k'}, f_{k'+1})$. Hence, because of the recurrence relation, we will have $f_{k+2} = f_{k'+2}$, $f_{k+3} = f_{k'+3}$, $\dots$ $f_{k+m} = f_{k'+m}$ for every $m \geq 0$. Taking $a = k' - k$, $b = k$ we get that the sequence $f_{an+b}$ is constant and equal to $f_k$, and the desired result follows.

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**Problem A6.** Let $a, b, c$ three positive real numbers prove:

$$\sqrt{a^2+1}+\sqrt{b^2+4}+\sqrt{c^2+9} \geq 2\sqrt{3}\sqrt{a+b+c}\,.$$

- *Answer:* Consider the complex numbers $u = a + i$, $v = b + 2i$, $w = c + 3i$. By the triangle inequality: $|u| + |v| + |w| \geq |u + v + w|$, i.e.:

$$\sqrt{a^2+1}+\sqrt{b^2+4}+\sqrt{c^2+9} \geq \sqrt{(x+y+z)^2+36}\,.$$

By the AM-GM inequality:

$$\frac{(x+y+z)^2+36}{2} \geq \sqrt{36(x+y+z)^2} = 6(x+y+z)\,.$$

Hence:

$$\sqrt{a^2+1}+\sqrt{b^2+4}+\sqrt{c^2+9} \geq \sqrt{12(x+y+z)} = 2\sqrt{3}\sqrt{a+b+c}\,.$$

□