Problem A1. Prove that the following equation has no solutions in positive integers:

\[ 8x^4 + 4y^4 + 2z^4 = t^4. \]

(Hint: \( t \) must be an even integer.)

- **Answer:** Since the left hand side is even then \( t \) must be even, \( t = 2t' \), hence:

\[ 8x^4 + 4y^4 + 2z^4 = 16t'^4, \]

and simplifying by 2:

\[ 4x^4 + 2y^4 + z^4 = 8t'^4, \]

Now all terms different from \( z^4 \) are even, hence \( z \) must be even: \( z = 2z' \), hence:

\[ 4x^4 + 2y^4 + 16z'^4 = 8t'^4. \]

Simplifying by 2 again we get

\[ 2x^4 + y^4 + 8z'^4 = 4t'^4. \]

A similar reasoning shows that \( y \) must be even, \( y = 2y' \), and after simplifying we get:

\[ x^4 + 8y'^4 + 4z'^4 = 2t'^4. \]

Next we do the same for \( x \) so we have \( x = 2x' \), and after simplifying:

\[ 8x'^4 + 4y'^4 + 2z'^4 = t'^4. \]

This shows that given a solution in positive integers \( (x, y, z, t) \) then there is another solution \( (x', y', z', t') \) in positive integers with \( x' = x/2 < x, y' = y/2 < y, z' = z/2 < z, t' = t/2 \). Repeating the reasoning we get an infinite sequence of positive solutions of the form \( (x/2^k, y/2^k, z/2^k, t/2^k) \) for \( k \) arbitrarily large, but that is impossible because no integer is divisible by arbitrarily large powers of 2.

\[ \square \]
Problem A2. Let \(a_1, a_2, a_3, \ldots\) a strictly increasing sequence of positive integers, i.e., \(a_n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}\), and \(n < m \Rightarrow a_n < a_m\) for every \(m, n\). Find all strictly increasing functions \(f : \mathbb{Z}^+ \to \mathbb{Z}^+,\) where \(\mathbb{Z}^+ = \{1, 2, 3, \ldots\}\), such that \(f(a_n) \leq a_n\) for every \(n \in \mathbb{Z}^+\).

- Answer: The only solution is \(f(n) = n\) for every \(n \in \mathbb{Z}^+\).

In fact, since \(f\) is strictly increasing then \(f(n) < f(n + 1)\) for every \(n\), hence \(f(n) + 1 \leq f(n + 1)\), and by induction \(f(n) = f(1 + (n - 1)) \geq f(1) + n - 1\). By hypothesis we have \(f(a_1) \leq a_1\), hence \(f(1) + a_1 - 1 \leq f(a_1) \leq a_1\), and from here we get \(f(1) \leq 1\). But \(f(1) \in \mathbb{Z}^+\), hence \(f(1) = 1\), and \(f(n) \geq n\).

Finally we show that \(f(n) > n\) cannot hold for any \(n\). In fact if \(f(n) > n\) for some \(n\), consider any \(a_k > n\). Then \(f(a_k) = f(n + (a_k - n)) \geq f(n) + a_k - n > a_k\), contradicting the hypothesis \(f(a_n) \leq a_n\) for every \(n\).

So, we have \(f(n) \geq n\), and \(f(n) \neq n\), hence \(f(n) = n\) for every \(n\), Q.E.D.
Problem A3. Find the following limit:

$$L = \lim_{n \to \infty} \sqrt[n]{\prod_{k=1}^{n} \left(1 + \frac{k}{n}\right)^{1/(1+k/n)}}.$$

(Hint: Take the logarithm of the expression under the limit.)

- Answer: The logarithm of the expression inside the limit is

$$\sum_{k=1}^{n} \log \left(1 + \frac{k}{n}\right) \frac{1}{1 + \frac{k}{n}} \frac{1}{n}.$$

That is a Riemann sum for the integral

$$\int_{0}^{1} \frac{\log (1 + x)}{1 + x} \, dx = \left[\frac{1}{2} \log^2 (1 + x)\right]_{0}^{1} = \frac{\log^2 2}{2}.$$

Hence the limit is

$$L = e^{\log^2 2} = 2^{\log^2 \frac{2}{2}}.$$

□
Problem A4. A fair coin is tossed repeatedly. What is the expected number of times the coin will be tossed until getting two heads in a row for the first time?

- Answer: Let $E$ the expected number of tosses until getting two heads in a row for the first time.

Denote heads and tails as $H$ and $T$ respectively. The following three things can happen:

1. We get $T$ in the first toss, with probability $1/2$.
2. We get $HT$ in the first two tosses, with probability $1/4$.
3. We get $HH$ in the first two tosses, with probability $1/4$.

In cases (1) and (2), after getting any of those results, the expected number of additional tosses we must wait to get two heads in a row is again $E$, which yields $1 + E$ tosses with probability $1/2$, and $2 + E$ with probability $1/4$. In case (3) we end the sequence after only 2 tosses, and this happens with probability $1/4$. Hence:

$$E = \frac{1}{2} (1 + E) + \frac{1}{4} (2 + E) + \frac{1}{4} \cdot 2.$$ 

From here we get $E = 6$, \qed
Problem A5. Let $a_k, k = 1, 2, 3, \ldots$, be a sequence of strictly positive numbers of period $2N$. Show that

$$\sum_{j=1}^{2N} \frac{a_{N+j}}{a_j} \geq 2N.$$ 

- Answer: We will use $x + \frac{1}{x} \geq 2$ for every positive real number $x$. We have

$$\sum_{j=1}^{2N} \frac{a_{N+j}}{a_j} = \sum_{j=1}^{N} \frac{a_{N+j}}{a_j} + \sum_{j=1}^{N} \frac{a_{2N+j}}{a_{N+j}}$$

$$= \sum_{j=1}^{N} \frac{a_{N+j}}{a_j} + \sum_{j=1}^{N} \frac{a_j}{a_{N+j}} \quad (a_{2N+j} = a_j)$$

$$= \sum_{j=1}^{N} \left( \frac{a_{N+j}}{a_j} + \frac{a_j}{a_{N+j}} \right)$$

$$\geq \sum_{j=1}^{N} 2 = 2N.$$ 

\[\square\]
Problem A6. Given any positive integer $a$ consider the sequence $a_n = a^{an}$, $n = 1, 2, 3, \ldots$. Prove that regardless of the integer $a$ chosen, the rightmost digit of the decimal representation of $a_n$ remains constant.

- Answer: First note that if $a = 1$ then $a_n = 1$ for every $n$, and this case is trivial, so in the following we assume $a \geq 2$.

We must prove that $a_n$ is constant modulo 10.

Note that for any integer $a$, the sequence $a^k$ for $k = 1, 2, 3, \ldots$ is periodic modulo 10 with a period of 1, 2 or 4, with the rightmost digits following one of these patterns:

- $0 \to 0$
- $1 \to 1$
- $2 \to 4 \to 8 \to 6 \to 2$
- $3 \to 9 \to 7 \to 1 \to 3$
- $4 \to 6 \to 4$
- $5 \to 5$
- $6 \to 6$
- $7 \to 9 \to 3 \to 1 \to 7$
- $8 \to 4 \to 2 \to 6 \to 8$
- $9 \to 1 \to 9$

Hence $a^k$ modulo 10 depends only on the value of $k$ modulo 4, meaning that if $k \equiv k' \pmod{4}$ then $a^k \equiv a^{k'} \pmod{10}$.

Hence $a^{a^k}$ modulo 10, depends only on the value of $a^k$ modulo 4. But $a^k$ is eventually periodic modulo 4 with period 1 or 2, and patterns $0 \to 0$, $1 \to 1$, $2 \to 0 \to 0$, $3 \to 1 \to 3$ (all modulo 4). Furthermore note that we can drop “eventually” if $k \geq 2$, since the only case in which $a^k$ is not strictly periodic module 4 is at the starting point of the pattern $2 \to 0 \to 0$, more specifically, if $a \equiv 2 \pmod{4}$ then $a^1 \equiv 2 \pmod{4}$, and for $k \geq 2$ we have $a^k \equiv 0 \pmod{4}$. Hence for $k \geq 2$, $a^k$ modulo 4 depends only on the parity of $k$, meaning than if $k, k' \geq 2$ and $k$ and $k'$ have the same parity then $a^k \equiv a^{k'} \pmod{4}$.

Finally we have that the parity of $a^n$ is the same as the parity of $a$, hence the parity of $a^n$ remains constant. This fact (combined with $a^n \geq 2$) implies that $a^{a^n}$ modulo 4 remains constant. And this implies that $a^{a^n}$ modulo 10 remains constant. \qed