NORTHWESTERN UNIVERSITY Thursday, Oct 18th, 2018

ANSWERS

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FALL 2018 NU PUTNAM SELECTION TEST

Problem A1. Suppose that a non-negative integer n is the sum of two triangular numbers

$$n = \frac{a^2 + a}{2} + \frac{b^2 + b}{2}$$

with (non-negative) integers a, b. Write $4n+1$ into the sum of two squares, i.e., $4n+1 = x^2+y^2$ with integers x, y. Express x and y in terms of a and b.

Show, conversely, that if $4n + 1$ is the sum of two squares, then n is the sum of two triangle numbers.

- *Answer:* We have

$$4n + 1 = 2a^2 + 2a + 2b^2 + 2b + 1 = (a + b + 1)^2 + (a - b)^2 \,,$$

so $x = a + b + 1$, $y = a - b$ is a solution.

Conversely, if $4n + 1 = x^2 + y^2$. Then x, y must be have different parities and the equations

$$a + b + 1 = x, \qquad a - b = y \,,$$

are solvable for integers a and b:

$$a = \frac{x + y - 1}{2}, \qquad b = \frac{x - y - 1}{2} \,.$$

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Problem A2. Prove that if m, n are positive integers such that $\sqrt{3} > \frac{m}{n}$, then $\sqrt{3} \geq \frac{\sqrt{m^2+2}}{n}$.

- *Answer:* We have

$$\sqrt{3} > \frac{m}{n} \Rightarrow 3m^2 > n^2 \Rightarrow 3m^2 \geq n^2 + 1\,.$$

We will prove next that in the last expression equality never holds. In fact, we have that $3m^2$ is a multiple of 3, and n^2 can only be 0 or 1 modulo 3, hence n^2+1 cannot be a multiple of 3. Consequently $3m^2 > n^2 + 1$, and since both sides are integers, $3m^2 \geq n^2 + 2$. From here the result follows.

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Note: A similar solution can be obtained reasoning modulo 4 using $n^2 \equiv 0 \text{ or } 1 \pmod 4$.

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Problem A3. Let $P(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$, with $a_i \in \mathbb{R}$, $i \in \{0, 1, \ldots, n-1\}$, be a polynomial with roots $x_1, x_2, \ldots, x_n \in \mathbb{R}$, and let $x_{i_0} = \max_{1 \le i \le n} x_i$. Prove that if $x \ge x_{i_0}$, then $P'(x) \ge n\sqrt[n]{(P(x))^{n-1}}$.

- *Answer:* First, factor the polynomial:

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_n)\,.$$

Note that the hypothesis $x \ge x_{i_0}$ implies that all the factors are positive. Next, differentiate:

$$P'(x) = P_1(x) + P_2(x) + \cdots + P_n(x)\,,$$

where $P_i(x) = P(x)/(x - x_i)$, $1 \le i \le n$. By the AM-GM inequality we have

$$\frac{1}{n}P'(n) = \frac{1}{n}\sum_{i=1}^{n} P_i(x) \ge \Big(\prod_{i=1}^{n} P_i(x)\Big)^{1/n}\,.$$

Also we have

$$\prod_{i=1}^{n} P_i = \frac{(P(x))^n}{P(x)} = (P(x))^{n-1}\,,$$

and the result follows.

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Problem A4. Define two sequences recursively as follows: $a_1 = b_1 = 1$, and for $n \geq 1$, $a_{n+1} = 2^{a_n}$, $b_{n+1} = 3^{b_n}$. Note that $a_3 = 4 > 1 = b_1^2$, $a_4 = 16 > 9 = b_2^2$, $a_5 = 65536 > 729 = b_3^2$. Prove that in general $a_{n+2} > b_n^2$ for every $n \geq 1$.

- *Answer:* We will prove it by induction. We have already checked the cases $n = 1, 2, 3$: $a_3 = 4 > 1 = b_1^2$, $a_4 = 16 > 9 = b_2^2$, $a_5 = 65536 > 729 = b_3^2$.

Next, assume $n \geq 3$, and $a_{n+2} > b_n^2$. Taking into account that $9 < 2^{27} = 2^{b_3} \leq 2^{b_n}$, and using the induction hypothesis, we get

$$b_{n+1}^2 = (3^{b_n})^2 = 3^{2b_n} = (3^2)^{b_n} = 9^{b_n} < (2^{b_3})^{b_n} \leq (2^{b_n})^{b_n} = 2^{b_n^2} < 2^{a_{n+2}} = a_{n+3} \,.$$

This completes the induction step, and proves the result.

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Problem A5. Let P_n be a regular n-gon ($n \geq 3$) inscribed in a unit circle (radius 1), and let $v_1, \dots, v_n$ be its vertices. Let d_{jk} be the distance between vertices v_j and v_k. Find the sum of the squares of distances between all pairs of vertices $S = \sum_{1 \leq j < k \leq n} d_{jk}^2$.

- *Answer:* The answer is $S = n^2$.

We can solve this problem using complex numbers. The vertices of the regular n-gon can be assumed to be the complex points $z_j = e^{2\pi ij/n}$, $j = 0, \dots, n-1$, and the square of the distance from z_j to z_k is

$$\begin{aligned} |z_k - z_j|^2 = |e^{2\pi ik/n} - e^{2\pi ij/n}|^2 &= (e^{2\pi k/n} - e^{2\pi ij/n})(e^{-2\pi ik/n} - e^{-2\pi ij/n}) \\ &= 1 - e^{2\pi i(k-j)/n} - e^{2\pi i(j-k)/n} + 1 \\ &= 2 - 2\Re\{e^{2\pi i(k-j)/n}\}\,, \end{aligned}$$

where $\Re\{z\}$ = real part of z.

The sum of the squares of the distances from each vertex j to all other vertices is

$$\begin{aligned} \sum_{k=0}^{n-1} (2 - 2\Re\{e^{2\pi i(k-j)/n}\}) &= 2n - 2\Re\Big\{\sum_{\ell=0}^{n-1} e^{2\pi i\ell/n}\Big\} \\ &= 2n - 2\frac{e^{2\pi i} - 1}{e^{2\pi i/n} - 1} = 2n\,. \end{aligned}$$

In that sum we include the distance from z_j to itself, which is zero, so it does not affect the final result, but makes the computation easier.

Since there are n vertices z_j ($j = 0, \dots, n-1$), multiplying by n we get the sum of the squares of distances, but with each pair of vertices counted twice, so we still must divide the product by 2 and we finally get $S = n^2$.

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Problem A6. Suppose that $a_n > 0$ and $\sum a_n$ diverges. Let $s_n = \sum_{i=1}^{n} a_i$ be the partial sums. For which positive values of p does the series

$$\sum_{n=1}^{\infty} \frac{a_n}{s_n^p}$$

converge?

- *Answer:* The series converges if and only if $p > 1$. Suppose that $p > 1$. Then for $n \geq 2$,

$$\frac{a_n}{s_n^p} = \frac{s_n - s_{n-1}}{s_n^p} = \frac{1}{s_n^p} \int_{s_{n-1}}^{s_n} dx \leq \int_{s_{n-1}}^{s_n} \frac{dx}{x^p}\,.$$

Hence,

$$\sum_{n=2}^{\infty} \frac{a_n}{s_n^p} \leq \sum_{n=2}^{\infty} \int_{s_{n-1}}^{s_n} \frac{dx}{x^p} = \int_{a_1}^{\infty} \frac{dx}{x^p} < \infty\,.$$

This shows that the series converges.

Now suppose that $0 < p \leq 1$. There is an index ℓ such that $s_\ell > 1$. We have for $n \geq \ell$,

$$\frac{a_n}{s_n^p} \geq \frac{a_n}{s_n} \stackrel{\text{def}}{=} b_n\,.$$

From $1 - b_n = s_{n-1}/s_n$ and $s_m \to \infty$ as $m \to$ we have

$$\prod_{n=\ell+1}^{\infty} (1 - b_n) = \lim_{m\to\infty} \frac{s_\ell}{s_m} = 0\,.$$

This shows that the infinite product diverges, hence the sum $\sum_{n=\ell+1}^{\infty} b_n$ also diverges by the convergence theorem for infinite products.[1] This shows that the series diverges for $0 < p \leq 1$.

□

[1] For real numbers with $0 \leq a_n \leq 1$, the infinite product $\prod_{n=1}^{\infty}(1-a_n)$ converges to a nonzero real number if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.

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Problem A7. Let $f : [a, b] \to \mathbb{R}$ be a continuous function defined on a finite interval $[a, b]$ such that it is twice continuous differentiable and $f(a) = f(b) = 0$.

Show that there is a constant C independent of a, b, and f such that

$$\int_a^b |f(x)|\, dx \leq C\, \|f''\|_\infty (b-a)^3 \,.$$

Here $\|f''\|_\infty = \sup_{x\in[a,b]} |f''(x)|$.

(Note: Next problem asks you to prove this same inequality for a specific value of C).

- *Answer:*

Solution 1. Because $f(a) = f(b)$, by Rolle's theorem there is some point $c \in (a, b)$ such that $f'(c) = 0$. Hence:

$$\begin{aligned} f(x) = f(x) - f(a) &= \int_a^x f'(y)\, dy \,, \\ f'(y) = f'(y) - f'(c) &= \int_c^y f''(z)\, dz \\ |f(x)| = \left| \int_a^x \int_c^y f''(z)\, dy\, dz \right| &\leq \int_a^x \int_c^y |f''(z)|\, dy\, dz \\ &\leq \|f''\|_\infty (x-a)(y-c) \leq \|f''\|_\infty (x-a)(b-a) \,. \end{aligned}$$

Integrating respect to x we get the result with $C = 1/2$. □

Solution 2. Given any point $x \in (a, b)$, by the Mean Value Theorem there is some point $x_1 \in (a, x)$ such that

$$f'(x_1) = \frac{f(x) - f(a)}{x-a} = \frac{f(x)}{x-a} \,,$$

and also there is a point $x_2 \in (x, b)$ such that

$$f'(x_2) = \frac{f(x) - f(b)}{x-b} = \frac{f(x)}{x-b} \,.$$

Next, applying the MVT to f', there is some point $c \in (x_1, x_2)$ such that

$$f''(c) = \frac{f'(x_2) - f'(x_1)}{x_2 - x_1} = \frac{f(x)}{x_2 - x_1}\left(\frac{1}{x-b} - \frac{1}{x-a}\right) = \frac{f(x)(b-a)}{(x_2-x_1)(x-b)(x-a)} \,,$$

hence

$$|f(x)| = \frac{|f''(c)| \cdot |x_2 - x_1| \cdot (b-x)(x-a)}{b-a} \leq \|f''\|_\infty (x-a)(b-x) \,,$$

Integrating between a and b we get the result with $C = 1/6$. □

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Problem A8. Let $f : [a,b] \to \mathbb{R}$ be a continuous function defined on a finite interval $[a,b]$ such that it is twice continuous differentiable and $f(a) = f(b) = 0$.

(a) Show that for every $x \in (a,b)$:

$$|f(x)| \le \|f''\|_\infty \cdot \frac{(b-x)(x-a)}{2}\,.$$

Here $\|f''\|_\infty = \sup_{x\in[a,b]} |f''(x)|$.

(b) Show that

$$\int_a^b |f(x)|\,dx \le \frac{1}{12}\|f''\|_\infty (b-a)^3\,.$$

(Note: This is the same as the previous problem, but now we require $C = 1/12$.)

- *Answer:* For a fixed $x \in [a,b]$ consider the function

$$h(y) = (b-y)(y-a)f(x) - (b-x)(x-a)f(y)$$

Then $h(a) = h(b) = h(x) = 0$. By Rolle's theorem there is a point $x_1 \in (a,x)$ such that $h'(x_1) = 0$, also there is a point $x_2 \in (x,b)$ such that $h'(x_2) = 0$, and applying Rolle's theorem again this time to h' we get that there is a point $c \in (x_1,x_2)$ such that $h''(c) = 0$.

We have

$$0 = h''(c) = -2f(x) - (b-x)(x-a)f''(c)\,.$$

Hence $2f(x) = -(b-x)(x-a)f''(c)$, and from here we get : $|f(x)| \le \|f''\|_\infty (b-x)(x-a)/2$. This shows (a). Integrating we obtain (b).

Note: The constant $1/12$ is best possible, as shown by the function $f(x) = (b-x)(x-a)$, for which the inequality becomes equality.

$\square$