Problem A1. Suppose that a non-negative integer \( n \) is the sum of two triangular numbers

\[
n = \frac{a^2 + a}{2} + \frac{b^2 + b}{2}
\]

with (non-negative) integers \( a, b \). Write \( 4n+1 \) into the sum of two squares, i.e., \( 4n+1 = x^2 + y^2 \) with integers \( x, y \). Express \( x \) and \( y \) in terms of \( a \) and \( b \).

Show, conversely, that if \( 4n+1 \) is the sum of two squares, then \( n \) is the sum of two triangle numbers.
Problem A2. Prove that if $m, n$ are positive integers such that $\sqrt{3} > \frac{m}{n}$, then $\sqrt{3} \geq \frac{\sqrt{m^2 + 2}}{n}$. 
Problem A3. Let \( P(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n \), with \( a_i \in \mathbb{R}, i \in \{0, 1, \ldots, n-1\} \), be a polynomial with roots \( x_1, x_2, \ldots, x_n \in \mathbb{R} \), and let \( x_{i_0} = \max_{1 \leq i \leq n} x_i \). Prove that if \( x \geq x_{i_0} \), then \( P'(x) \geq n \sqrt[n]{(P(x))^{n-1}} \).
Problem A4. Define two sequences recursively as follows: $a_1 = b_1 = 1$, and for $n \geq 1$, $a_{n+1} = 2^{a_n}$, $b_{n+1} = 3^{b_n}$. Note that $a_3 = 4 > 1 = b_1^2$, $a_4 = 16 > 9 = b_2^2$, $a_5 = 65536 > 729 = b_3^2$. Prove that in general $a_{n+2} > b_n^2$ for every $n \geq 1$. 
Problem A5. Let $P_n$ be a regular $n$-gon ($n \geq 3$) inscribed in a unit circle (radius 1), and let $v_1, \ldots, v_n$ be its vertices. Let $d_{jk}$ be the distance between vertices $v_j$ and $v_k$. Find the sum of the squares of distances between all pairs of vertices $S = \sum_{1 \leq j < k \leq n} d_{jk}^2$. 
Problem A6. Suppose that $a_n > 0$ and $\sum a_n$ diverges. Let $s_n = \sum_{i=1}^{n} a_i$ be the partial sums. For which positive values of $p$ does the series

$$\sum_{n=1}^{\infty} \frac{a_n}{s_n^p}$$

converge?
**Problem A7.** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function defined on a finite interval \([a, b]\) such that it is twice continuous differentiable and \( f(a) = f(b) = 0 \).

Show that there is a constant \( C \) independent of \( a, b, \) and \( f \) such that

\[
\int_a^b |f(x)| \, dx \leq C \|f''\|_{\infty} (b - a)^3.
\]

Here \( \|f''\|_{\infty} = \sup_{x \in [a,b]} |f''(x)| \).

(Note: Next problem asks you to prove this same inequality for a specific value of \( C \).)
Problem A8. Let $f : [a, b] \to \mathbb{R}$ be a continuous function defined on a finite interval $[a, b]$ such that it is twice continuous differentiable and $f(a) = f(b) = 0$.

(a) Show that for every $x \in (a, b)$:

$$|f(x)| \leq \|f''\|_{\infty} \cdot \frac{(b - x)(x - a)}{2}.$$ 

Here $\|f''\|_{\infty} = \sup_{x \in [a, b]} |f''(x)|$.

(b) Show that

$$\int_a^b |f(x)| \, dx \leq \frac{1}{12} \|f''\|_{\infty} (b - a)^3.$$ 

(Note: This is the same as the previous problem, but now we require $C = 1/12$.)