PRELIMINARY EXAM IN ALGEBRA FALL 2023

INSTRUCTIONS:

- Do **all** of the following problems.
- In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I

Do **all** of the following three problems.

(1) (a) For $\Bbbk$ a field, prove that the group B of upper triangular matrices in $\mathrm{GL}_n(\Bbbk)$ is solvable.

(b) Find a p-Sylow subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ and prove it is indeed p-Sylow.

(2) (a) Recall that an associative ring R is left (respectively, right) semisimple if R is a direct sum of simple left (respectively, right) R-modules. Prove that R is left semisimple if and only if it is right semisimple.

(b) Let $R = \mathbb{C}[G]$ be the group ring on a finite abelian group G. Prove that R is isomorphic as a ring to a product of fields $\mathbb{C}$.

(c) For G a finite group, prove that every finite-dimensional $\mathbb{C}[G]$-module V is isomorphic to a direct sum of simple $\mathbb{C}[G]$-modules.

(3) (a) Let $\iota : H \subset G$ be a subgroup of finite index. Define the transfer map on group homologies

$$\mathrm{tr} : \mathrm{H}_*(G, \mathbb{Z}) \to \mathrm{H}_*(H, \mathbb{Z}) ,$$

and prove that the resulting composite map $\iota_* \circ \mathrm{tr}$ is multiplication by $[G : H]$.

(b) Consider the semidirect product of groups $\mathbb{Z}^\times \ltimes \mathbb{Z}$, where the element $-1 \in \mathbb{Z}^\times$ acts on the additive group $\mathbb{Z}$ as multiplication. Calculate the group homology

$$\mathrm{H}_*(\mathbb{Z}^\times \ltimes \mathbb{Z}, \mathbb{F}_p)$$

for all odd primes p. Here the action of the group on $\mathbb{F}_p$ is trivial.

Part II

Do **all** of the following three problems.

(1) Let $\mathbb{k}$ be any field. Find all the solutions $f(x)$ of the system of congruences

$$f(x) \equiv x^2 + 1 \mod x^3 + x + 1;$$
$$f(x) \equiv x + 1 \mod x^2 - 1$$

in $\mathbb{k}[x]$.

(2) Write down all possible expressions for the minimal polynomial of any primitive root of unity of order 15 in the algebraic closure $\overline{\mathbb{k}}$ where

a) $\mathbb{k} = \mathbb{Q}$;

b) $\mathbb{k} = \mathbb{F}_2$.

(3) Show that

$$f(x) = x^5 - 21x^2 + 6$$

is irreducible over $\mathbb{Q}$. Compute the Galois group of its splitting field over $\mathbb{Q}$.

Part III

Do **all** of the following three problems.

(1) Let $\mathfrak{m} \subset R$ and $\mathfrak{n} \subset S$ be local Noetherian rings, and let $f : R \to S$ be a finite ring homomorphism such that:
  (a) $f(\mathfrak{m}) \subset \mathfrak{n}$ and the resulting map of residue fields $\kappa(\mathfrak{m}) \to \kappa(\mathfrak{n})$ is an isomorphism;
  (b) the map of Zariski cotangent spaces $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$ is surjective.

  Prove that $f : R \to S$ is surjective.

(2) Consider the ring $R = \mathbb{C}[x, y]/(y^2 - x^3 + ax)$ for $a \in \mathbb{C}$. For each value of a determine:
  (a) the Krull dimension Kr.dim R;
  (b) whether for every maximal ideal $\mathfrak{m} \subset R$ there is equality

$$\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = \text{Kr.dim } R_{\mathfrak{m}}$$

  where $\kappa(\mathfrak{m})$ is the residue field at $\mathfrak{m}$. (I.e., determine whether $R_{\mathfrak{m}}$ is a regular local ring.)

(3) Let R be a Noetherian ring and M a finitely-generated R-module. Give a complete proof of the following: There exists a finite sequence of inclusions of R-modules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

  such that for each $0 \leq i < n$ there exists an isomorphism of R-modules

$$M_{i+1}/M_i \cong R/\mathfrak{p}_i$$

  for some $\mathfrak{p}_i \subset R$ a prime ideal.