PRELIMINARY EXAM IN ALGEBRA SPRING 2023

INSTRUCTIONS: (1) There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three, then indicate which you would like graded; otherwise we will grade the first three you attempt in each section.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Autumn

Do **three** of the following five problems.

(1) (a) Let S be a finite set acted on by a finite group H of order p^n, where p is prime. Prove that the number of elements of S and of the fixed-point set S^H are equal mod p:
$$|S^H| = |S| \bmod p$$
(b) Prove Cauchy's Theorem: If G is a finite group with order divisible by a prime p, then G contains an element of order p.

(2) Let G be a finite group of order 40. Prove that G is not simple.

(3) Consider the symmetric group Σ_3, of automorphisms of a 3-element set, and the cyclic groups C_n of order n.
(a) What are the Sylow p-subgroups of Σ_3, for all p?
(b) Calculate the group cohomology $\mathrm{H}^*(C_2, \mathbb{F}_2)$, for $* = 0, 1$.
(c) Calculate the group cohomology $\mathrm{H}^*(\Sigma_3, \mathbb{F}_2)$, for $* = 0, 1$.

(4) The infinite dihedral group D is the semidirect product $C_2 \ltimes \mathbb{Z}$, where $\sigma \in C_2$ acts by multiplication by -1. Calculate the first group homology $\mathrm{H}_1(D, \mathbb{Z})$.

(5) Let $\mathcal{C}$ be a category. The Yoneda functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sets})$ sends an object c to the contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-, c)$. Prove that the Yoneda functor is fully faithful.

Part II. Winter

Do **three** of the following five problems.

(1) Let $R = \mathbb{Z}[\sqrt{-2}]$. Let $N(z) = z\bar{z}$ for any complex number z. Show that for any $a \neq 0$ and b in R there are q and r in R such that $b = aq + r$ and $N(r) < N(a)$. From this, deduce that R is a PID. Find such q and r for $b = 10 + 3\sqrt{-2}$ and $a = 2 - 3\sqrt{-2}$.

(2) Classify up to similarity 4×4 matrices A over a field K satisfying

$$(A^2 + A + 1)(A - 1)^2 = 0$$

where
(a) $K = \mathbb{C}$
(b) $K = \mathbb{Q}$
(c) $K = \mathbb{F}_2$

(3) Show that for any a and b in a finite field F the equation

$$(X^3 + aX + b)(X^2 + 27b^2 + 4a^3) = 0$$

has a solution in F.

(4) Find the degree of the splitting field of $X^{20} - 1$ over K where
(a) $K = \mathbb{Q}$
(b) $K = \mathbb{F}_3$

(5) Let K be the splitting field of $X^4 - 10$ over $\mathbb{Q}$. Find the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ and all the subfields of K.

Part III. Spring

Do **three** of the following five problems.

(1) Let $K[x_1, x_2, \ldots]$ be a polynomial ring in infinitely-many variables over a field K, and let I be the ideal $(x_1, x_2, \ldots)$. Set

$$R = K[x_1, x_2, \ldots]/I^2 .$$

(a) What is topological space $\mathrm{Spec}(R)$?
(b) Is R Artinian? Noetherian?
(c) Compute the Krull dimension of R.

(2) Let R be a commutative ring with

$$M \xrightarrow{\varphi} M$$

an R-linear endomorphism of a finitely-generated R-module M. Prove that if φ is surjective, then it is also injective. (Hint: Recall Nakayama's Lemma, and note that φ gives M the structure of a module over the polynomial ring $R[x]$.)

(3) Let K be a field. Prove that the polynomial ring $K[x_1, \ldots, x_n]$ is a Noetherian ring.

(4) Consider the ring

$$R = \mathbb{R}[x, y, z, w]/(xy - zw, z^3) .$$

(a) Construct a finite ring homomorphism $A \to R$ where A is a polynomial algebra.
(b) Compute the Krull dimension of R. Construct a chain of prime ideals of length $\dim(R)$.
(c) Compute the Krull dimension of $R[x^{-1}]$. Construct a chain of prime ideals of length $\dim(R[x^{-1}])$.

(5) Let K be a field. Give a complete proof that there is an equivalence of categories

$$\mathrm{Mod}_K \cong \mathrm{Mod}_{M_n(K)}$$

between K-modules and modules over the ring $M_n(K)$ of $(n \times n)$-matrices over K.