Algebra Preliminary Examination

Northwestern University, September 2017

Do all of the following questions. Each question is worth 0.5 points.

Question 1. Let α be a root of $X^6 + X^3 + 1$. Find all homomorphisms $\mathbb{Q}(\alpha) \to \mathbb{C}$ of fields.

Question 2. Let S_3 be the symmetric group on 3 elements, and k an algebraically closed field of characteristic zero.

1. Find all conjugacy classes of S_3.
2. Find the dimension and the multiplicity in the regular representation of all irreducible representations of S_3 over k.
3. Write down the character table of S_3 over k.

Question 3. Let K be a field and let $\mathrm{M}_3(K)$ denote the K-algebra of 3-by-3 matrices. Let B denote the subalgebra of $\mathrm{M}_3(K)$ of upper-triangular matrices. Determine whether B is semisimple.

Question 4. Put $R = \mathbb{F}_q[X, Y]/\langle X^q Y - XY^q \rangle$ where q is a power of prime. Let x, y be the image of X, Y in R, respectively. Show that for every $a \in \mathbb{F}_q$, R is not a finitely generated module over $\mathbb{F}_q[y - ax]$.

Question 5. Let G be a group and H a subgroup of finite index. Show that there exists a normal subgroup N of G contained in H and also of finite index.

Question 6. Let R be a commutative ring that is a finitely generated $\mathbb{Z}$-algebra. Let $\mathfrak{m}$ be a maximal ideal of R. Show that $R/\mathfrak{m}$ is a finite field.