PRELIMINARY EXAM IN ANALYSIS SPRING 2017

INSTRUCTIONS:

(1) There are **three** parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

Problem 1. (a) State the three convergence theorems of Lebesgue integration theory: (1) the monotone convergence theorem; (2) Fatou's lemma; (3) the dominated convergence theorem. (b) Give an example for which the strict inequality holds in Fatou's lemma.

Problem 2. State Hölder's inequality and the Minkowski inequality. (b) Prove the following reverse Hölder's inequality. Let f and g be two strictly positive measurable functions on a measure space $(X, \mathscr{F}, \mu)$. Let p and q are two real numbers such that $0 < p < 1$ and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\|fg\|_1 \geq \|f\|_p \|g\|_q.$$

Note that here $-\infty < q < 0$ and for a nonzero real number r and a positive measurable function h,

$$\|h\|_r = \left(\int_X h^r \, d\mu \right)^{1/r}.$$

Problem 3. A family $\mathscr{S}$ of (Lebesgue) measurable functions on the unit interval $[0, 1]$ is said to be uniformly intetegrable if for any positive ϵ there is a positive λ such that

$$\int_E |f| \leq \epsilon$$

for all $f \in \mathscr{S}$ and all measruable sets $E \in \mathscr{F}$ such that (Lebesgue measure) $|E| \leq \lambda$. Show that $\mathscr{S}$ is uniformly integrable if and only if

$$\lim_{C \to \infty} \sup_{f \in \mathscr{S}} \int_{\{|f| \geq C\}} |f| = 0.$$

Problem 4. (a) Define the total variation $V(F)_a^b$ of a function F defined on a finite interval $[a, b]$. (b) Show that if f is an integrable function on a finite interval $[a, b]$ and

$$F(x) = \int_a^x f(t) \, dt.$$

Then

$$V(F)_a^b = \int_a^b |f(t)|\, dt.$$

You may assume that f is continuous to simplify your proof. (c) Let $F(x) = x\sin(1/x)$ for $0 < x \leq 1$ and $F(0) = 0$. Is F a function of bounded variation on $[0,1]$? Prove that your answer is correct.

Problem 5. Suppose that $f_n \in L^p(X, \mathscr{F}, \mu)$ with $1 \leq p < \infty$ and

$$\sum_{n=1}^{\infty} \|f_n\|_p < \infty.$$

Let

$$F_N(x) = \sum_{n=1}^{N} f_n(x).$$

Show that F_N converges in $L^p(X, \mathscr{F}, \mu)$.

Part II. Functional Analysis

Do **three** of the following five problems.

Problem 1. Suppose that $\mu(X) = 1$. Show that if $f \in L^\infty(X, \mu)$ then the function $p \mapsto ||f||_p$ is increasing and $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

Problem 2. Let $K(x,y)$ be a measurable function on $X \times Y$ where $(X, \mu), (Y, \nu)$ are two finite (or σ-finite) measure spaces. Let

$$T_K f(x) = \int_Y K(x,y) f(y)\, \nu(dy).$$

Prove that $T_K : L^p(Y, \nu) \to L^p(X, \mu)$ is bounded for all $1 \leqslant p \leqslant \infty$ if

$$\sup_x \int_X |K(x,y)|\, \nu(dy) < \infty \text{ and } \sup_y \int_X |K(x,y)|\, \mu(dx) \ < \infty,$$

and then

$$||T_K||_{L^p \to L^p} \leqslant \max\left\{\sup_x \int_X |K(x,y)|\, \nu(dy), \sup_y \int_X |K(x,y)| \mu(dx)\right\}.$$

Problem 3. Suppose $f \in L^1(\mathbb{R})$. Show that its Fourier transform $\hat{f}$ is in $C_0(\mathbb{R})$ (the space of continuous functions on $\mathbb{R}$ vanishing at infinity).

Problem 4. Define Hilbert-Schmidt operator $T : H \to H$ on an infinite dimensional Hilbert space H. Can a Hilbert-Schmidt operator $T : H \to H$ be surjective? Prove that your answer is correct.

Problem 5. (a) Define the Sobolev spaces $H^s(\mathbb{R}^n)$ where s is a non-negative real number. (b) Let $\mathbf{1}_Q$ be the characteristic function of the unit box $Q = [0,1] \times [0,1] \times \cdots \times [0,1]$ (n times). Find the set of s so that $\mathbf{1}_Q \in H^s(\mathbb{R}^n)$.

Part III. Complex Analysis

Do **three** of the following five problems.

Problem 1. Let $f : D \to D$ be a holomorphic map, where $D = D_1(0)$ is the unit disc in $\mathbb{C}$. Prove that for all $a \in D$ we have

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leqslant \frac{1}{1 - |a|^2}.$$

Hint: Recall that for any $\alpha \in D$,

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$$

is an automorphism of D that maps α to 0.

Problem 2. Let n be a positive integer, and C the boundary of the unit disc with counterclockwise orientation. (a) Compute

$$\int_C \left(z - \frac{1}{z}\right)^n \frac{dz}{z},$$

(b) Use the result in (a) to evaluate the real-variable integral

$$\int_0^{2\pi} \sin^n(x) dx.$$

Problem 3. Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic functions satisfying $|f(z)| \leqslant 2017|g(z)|$ for all $z \in \mathbb{C}$. Show that $f = cg$ for some constant $c \in \mathbb{C}$.

Problem 4. Let f be a holomorphic function defined in a neighborhood of the closed unit disc $\overline{D}$, satisfying $f(\frac{1}{3}) = f(-\frac{1}{3}) = 0$. Prove that

$$|f(0)| \leqslant \frac{1}{9} \sup_D |f|.$$

Hint: you may use Jensen's formula,

$$\log |f(0)| = \sum_{\ell=1}^{n} \log |z_\ell| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta,$$

where f is a holomorphic function defined on $\overline{D}$ with $f(0) \neq 0$ and $\{z_\ell\}$ are the zeros of f in the interior of the unit disc.

Problem 5. Let $\Lambda, \Lambda' \subset \mathbb{C}$ be two lattices generated by $\{1, \tau\}$ and $\{1, \tau'\}$ respectively, where $\tau, \tau' \in \mathbb{H}$. Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ and $\pi' : \mathbb{C} \to \mathbb{C}/\Lambda'$ be the quotient maps. Suppose that $F : \mathbb{C} \to \mathbb{C}$ is a holomorphic function which descends to a map $G : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ (i.e. $\pi' \circ F = G \circ \pi$). Show that we must have $F(z) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$ with $\alpha\Lambda \subset \Lambda'$.