Geometry/Topology Exam

Student Id:_________________________

Instructions: Complete each problem, show your work in detail. Theorems which are used or quoted must be stated explicitly.

Score	
1	
2	
3	
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Total	

Question 1. Let X be a connected three-dimensional topological manifold and $x, y \in X$ be two distinct points. Show that the inclusion $X \backslash y \subset X$ induces an isomorphism $\pi_1(X \backslash y, x) \to \pi_1(X, x)$.

Question 2.

a Show that every continuous map $S^2 \to S^1$ is homotopy equivalent to the constant map.

b Show that every principal circle bundle over S^3 is trivial.

Question 3. Let (M^n, g) be a complete Riemannian manifold with $\gamma : [0, d] \to M^n$ a minimizing geodesic connecting x and y:

a Show for each $t < d$ that $\gamma : [0, t] \to M$ is a *minimizing* geodesic.

b Show for each $t < d$ that $\gamma : [0, t] \to M$ is the *unique* minimizing geodesic from x to $\gamma(t)$.

c Give an example of M and γ showing $\gamma : [0, d] \to M$ may not be a unique minimizing geodesic.

Question 4. Let (M^n, g) be a complete manifold with $p \in M$ and $d_p(x) = d(p, x)$ the distance function from p, so that $|\nabla d_p| = 1$.

a For any $x \in M$ such that $d_p(x)$ is smooth near x,show that the hessian satisfies $\nabla^2 d_p(\nabla d_p, X) = 0$ for any $X \in T_x M$.

b Show that if A is $k \times k$ matrix then $|A|^2 \geq \frac{1}{k}(trA)^2$, where trA is the trace of A and $|A|^2 = \sum_{ij} A_{ij}^2$ is its norm squared..

c Let $\gamma : [0, \ell) \to M$ be a minimizing geodesic from p and define $m(t) \equiv \Delta d_p(\gamma(t))$. Show for $t \in (0, \ell)$ that $\frac{d}{dt} m(t) \leq -\frac{1}{n-1} m^2(t) - Rc(\dot{\gamma}, \dot{\gamma})$, where $\Delta = g^{ij} \nabla_i \nabla_j$ is the laplacian and Rc is the Ricci curvature of the manifold.

d Conclude that if $Rc \geq 0$ then *whenever* d_p *is smooth* we have $\Delta d_p \leq \frac{n-1}{d_p}$. (Indeed, this holds in the barrier or distributional sense on all of M, but you don't need to prove this)

Question 5. Let M be a compact connected three-dimensional manifold without boundary.

a Prove that the Euler characteric $\chi(M)$ is zero, even if M is not orientable (you may use the fact that $\chi(M) = \sum_{i=0}^{3} (-1)^i dim(H_i(M, F))$ for any field F. That is, the right hand side is independent of F).

b Assume M is non-orientable, prove that $dim(H_1(M, \mathbb{Q})) > 0$ (Hint: Compute $\chi(M)$).

c Assume M is non-orientable, prove that $H_1(M, \mathbb{Z})$ is an infinite group.

Question 6. a Describe a CW complex whose underling space is $\mathbb{RP}^n$.

b Following the definition of cell cohomology, compute $H^k(\mathbb{RP}^4, \mathbb{Z})$ for all k.