Geometry/Topology Exam

Student Id:________________________

Instructions: Complete each problem, show your work in detail. Theorems which are used or quoted must be stated explicitly.

Score	
1	
2	
3	
4	
5	
6	
Total	

Question 1. Which of the following subspaces of $\mathbb{R}^2$ is a topological manifold? Justify

a The graph of the function $y = |x|$;

b The union of the coordinate axes.

Question 2. Calculate the fundamental group of a 'flat tire' (a two-dimensional torus with a small disk removed from it).

Question 3. Let $f(r) > 0$ be smooth with $f(0) = 0$ and $\frac{d}{dr} f(0) = 1$, and consider the warped product metric on $\mathbb{R}^2$ given by $g_{ij} = dr^2 + f^2(r) g_{S^1}$. Consider polar coordinates $(x^1, x^2) = (r, \theta)$:

- In polar coordinates compute the Levi-Civita connection $\Gamma_{ij}^k = \frac{1}{2} g^{sk} \big(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij} \big)$.
- In polar coordinates write the geodesic equation of a curve $\gamma(t) = (\gamma_r(t), \gamma_\theta(t))$.
- For each $\theta \in S^1$ show $\gamma(t) = (t, \theta)$ solves the geodesic equation.
- Prove the injectivity radius at the origin satisfies $inj(0,0) = \infty$. That is, show the previous geodesics are *globally* minimizing for $t \in [0, \infty)$.

Question 4. Show the following:

a Let (M^n, g) be a compact Riemannian manifold and let $f : M \to \mathbb{R}$ be a harmonic function $\Delta f = 0$. Show that f is a constant.

b Let (M^n, g) be a compact oriented Riemannian manifold and let $\omega \in \Omega^n(M)$ be a hodge harmonic form $(d^* d + dd^*)\omega = 0$. Show that ω is a multiple of the volume form.

Question 5. Let A, B be path connected open (nonempty) subsets of $\mathbb{S}^n$ so that $A \cup B = \mathbb{S}^n$.

a If $n \geq 2$, prove $A \cap B$ is path connected.

b Is the conclusion still true for $n = 1$?

Question 6. Compute the relative homology group $H_*(X, A; \mathbb{Z})$ for the following pairs:

a $X = \mathbb{S}^2$, A is the equator.

b X is the Möbius band, A is the boundary.